A Proof-Writing Instructional Design Model

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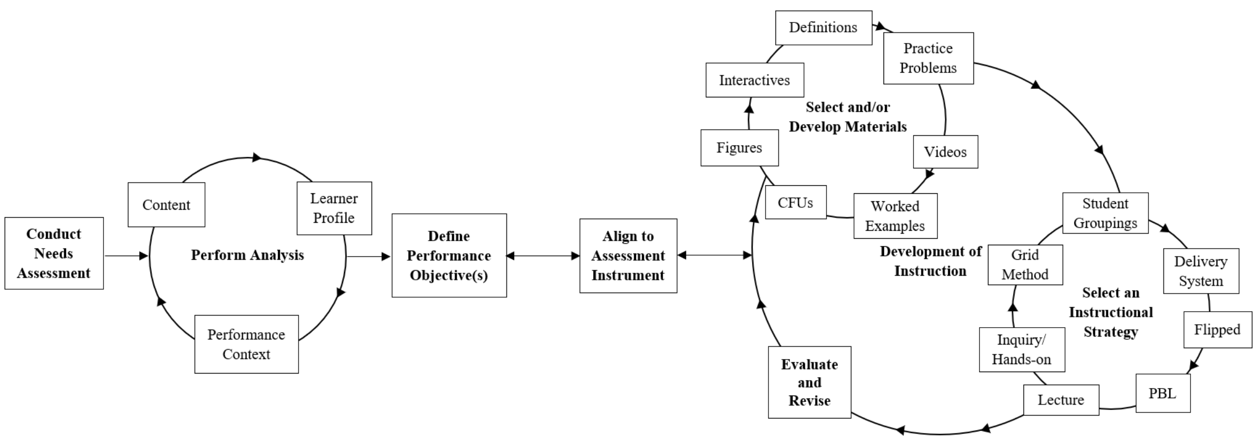
Proof-writing is one of the fundamental mathematical skills students learn in high school Geometry classes. Much like an instructional design model, proof-writing is a systematic, iterative process. Learners analyze given information and figures to develop a series of reasons and statements supported by theorems, postulates, and corollaries to justify the conclusion statement, with learners evaluating their logic and revising as needed throughout the process. As a fundamental part of geometric thinking, learners are often introduced to proof-writing through the use of algebraic proofs which are used to justify how to solve an algebraic equation. As learners hone their skills and build their understanding of the purpose and procedure of writing a proof, additional theorems, postulates, and corollaries are gradually introduced and the content of the proofs shifts from algebraic to geometric. As students learn about different geometric concepts, such as triangle congruence, quadrilaterals, and similar figures, learners are able to write increasingly complex proofs which incorporate both newly and previously learned postulates and theorems. The proof-writing process is a highly iterative process in which the writer quickly receives feedback about the validity of their reasoning as they follow through with their logic and are either able to continue to another statement or reach a dead-end that requires reevaluation of previous logic.

Since proof-writing is not confined to a specific topic within the Geometry curriculum but is instead a key and wide-spread application of geometric concepts, the instructional design model must provide the designer opportunity for choice based on the learners' needs, the specific content standards, and the performance context for the goal, which varies from constructing a paragraph, two-column, or flow chart proof to critiquing a provided proof to using geometric theorems, postulates, and axioms in additional applications depending on the content. Additionally, much of the instructional design is done by the instructor who will also teach the lesson, not a removed instructional designer. The unique challenges classroom instructors who are designing instruction and executing the planned instruction face were not taken into consideration in the Dick & Carey (1990) instructional design model. In order to address the specific needs of instructor-designers, the Proof-Writing Instructional Design Model (see Figure 1) was developed. While the model was developed from the Dick & Carey model, the model differs in the analysis of the needs assessment, the development of the assessment instrument, and the process of instructional material and strategy selection, evaluation, and revision.

Figure 1

Proof-Writing Instructional Design Model



Note. Individual sections of the model are included in later parts of the paper.

Purpose & Key Considerations

According to National Council of Teachers of Mathematics (NCTM), a three-year pathway of Algebra I, Geometry, and Algebra II is the general sequence of mathematics courses "offered by more than 90 percent of high schools to students in the United States" (2018, p. 1). In Ohio, high school students are required to have four units of high school mathematics in order to graduate, although the only math course required for graduation is Algebra II (Ohio Department of Education (ODOE), 2019a). Additionally, students are required to earn at least 18 points from seven end-of-course (EOC) exams, two of which are math tests: Algebra I and Geometry (ODOE, 2019b). Students earn points toward graduation depending upon their performance on the EOC exams (ODOE, 2019b). Since students' Geometry success impacts their ability to graduate, special attention is needed when designing instruction.

The Ohio Learning Standards for Mathematics state "proofs in high school mathematics should not be limited to geometry. Mathematically proficient high school students employ multiple proof methods, including algebraic derivations, proofs using coordinates, and proofs based on geometric transformations, including symmetries" (ODOE, 2018, p. 7). The Common Core State Standards Initiative, which is the basis for the Ohio Learning Standards for Mathematics, argues that the purpose of the high school Geometry standards and curriculum is "to formalize [students'] geometry experiences from elementary and middle school, using more precise definitions and developing careful proofs" (2021, para. 3). Bass (2015) described five phases that make up the process in which a learner acquires knowledge: exploration, discovery, conjecture, proof, and certification. Proof-writing, as defined by NCTM, "involves determining when and why a conjecture does or does not hold by using deductive reasoning – applying general theorems or statistical methods to specific instances" (2018, p. 39). In the context of Bass's five phases, proof-writing allows the learner to "generalize discoveries and deepen their understanding enough to help them remember an important claim and how to apply it" (NCTM, 2018, p. 39). As learners write proofs, justify their reasoning, and critique their own and peers' thinking, they are able to build their understanding of the applications of geometric concepts. Designing meaningful, engaging proof-writing instruction is key to ensuring students' success in Geometry and thus their ability to graduate. Since proof-writing is an integral part of Geometry and is part of all aspects of the Geometry curriculum, a proof-writing instructional design model can be used throughout the design of instruction.

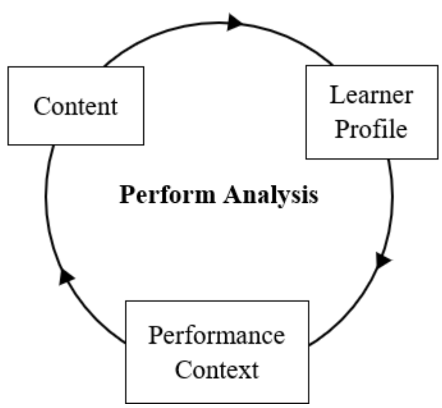
Components of the Model

Needs Assessment & Analysis

Conducting a needs assessment allows an instructional designer to understand the actual status of learners in comparison to the desired status (Dick et al., 2015). Dick et al. (2015) use the formula, “desired status – actual status = need” (p. 23) to determine if a gap exists for learners. Based on the needs assessment, an analysis (see Figure 2) is conducted so that the designer can develop a learner profile, determine the performance context, and analyze the instructional goal and content. The purpose of this analysis is to give the instructional designer the information necessary to design meaningful and effective instruction.

Figure 2

Areas of Analysis in Proof-Writing Instructional Design Model



In order to design effective, meaningful proof-writing instruction, the instructional designer must evaluate three components: the content, the learners, and the performance context. The designer conducts an instructional analysis to “identify the relevant steps for performing a goal and the subordinate skills required for a student to achieve the goal” (Dick et al., 2015, p. 42). The designer needs to understand the content because the application of proofs varies depending on the content being taught, both within an instructional unit and within the course itself. Many proofs depend upon previously taught theorems and postulates to further learners' thinking and to allow learners to make deep connection between content areas, but if the prior skills have not been addressed, modifications to instruction are necessary to ensure student success. As an instructor-designer, classroom instructors are subject matter experts who are able to analyze the content without the need for an additional expert. The content directs the instruction by describing what knowledge and skills learners should acquire by the end of the instructional unit.

Due to the importance of previous content knowledge in the proof-writing process, the instructional designer must also know about the learners in order to best address learners' needs. Dick et al. (2015) caution that a designer must "be clear about exactly who the learners will be rather than making vague statements or allusions to groups of learners" (p. 26). In order to understand the needs of the learners, the instructional designer conducts a learner analysis. During a learner analysis, an instructional designer collects information about learners' entry skills, prior knowledge, learning preferences and attitudes, and other group characteristics (Dick et al., 2015). As an instructor-designer, a classroom instructor benefits from the fact that they actually know and interact with the learners – the learner analysis is not about unknown learners, but about learners that the instructor has personal contact with. Since successful proof-writing depends upon a learners' previously learned theorems and postulates as well as newly learned theorems and postulates, learners' entry skills and prior knowledge are particularly important for instructional designers. Entry skills are the prerequisite skills learners should have before instruction, such as the previously learned theorems and postulates (Dick et al., 2015). Prior knowledge is "what learners already know about the topic to be taught" (Dick et al., 2015, p. 97), which is important because many of the high school Geometry concepts were previously introduced in elementary and middle school math classes.

The final component of the analysis is the performance context, which includes the situation where the skills will be used and any tools or aids that learners will be able to use (Dick et al., 2015). Since Geometry has an EOC state test, the performance context is extremely important in designing effective instruction. During their EOC exam, students receive a test reference sheet, which gives common conversions, trigonometry formulas, volume and area formulas, and other specific math formulas. Students also have access to a Desmos graphing calculator in the testing software. While instruction may be designed to incorporate tools beyond those that are accessible during the EOC exam, it is important for the instructional designer to keep in mind to what learners will have access during the state EOC exam.

Performance Objective

After conducting the analysis, the instructional designer now has a learner profile, knows the performance context, and understands what steps and subskills are necessary for learners to achieve the goal of the instruction. From this, the instructional designer writes the performance objectives that are the necessary steps for learners to master the content. Performance objectives not only provide direction for the instructional designer to develop the instruction, they also provide guidance for the learner (Dick et al., 2015). Dick et al. (2015) note, “knowledge of intended outcomes aids students in linking new knowledge and skills to their current knowledge and experiences” (p. 118). For the instructional designer, articulating the performance objectives makes it clear what learners will be able to do at the end of the instruction, which allows for increased accountability and accuracy of instruction during the design process (Dick et al., 2015). As someone who is in the classroom and has a deep understanding of the content and its connects, instructor-designers are able to clearly identify and write performance objectives.

For Geometry courses, instructional goals are derived from the Ohio Learning Standards for Mathematics which provide the tested standards in which all learners must be able to demonstrate mastery. However, these standards are rarely the performance objectives taught during a lesson due to their broad nature. For example, standard G.CO.9 states "Prove and apply theorems about lines and angles" (ODOE, 2018, p. 17). This standard encompasses multiple proof-writing and related performance objectives that are necessary for student mastery of the instructional goal and standard, including knowledge of theorems, postulates, and corollaries (Vertical Angles Theorem, Corollary to the Alternate Interior Angle Theorem, etc.) as well as definitions (transversal, parallel lines, bisector, etc.) and figures (alternate interior angles, same-side interior angles, etc.). According to Dick et al. (2015), an instructional designer should use the instructional analysis of the content from the first step in the instructional design process to write an objective for each identified skill, as well as applicable entry skills based on the learner analysis. In the case of standard G.CO.9, performance objectives could be written for each of the theorems, postulates, and corollaries that learners are expected to be able to apply to proof-writing, as well as performance objectives for the proof-writing itself. These performance objectives are essential for determining the sequencing of the instruction necessary for learner success.

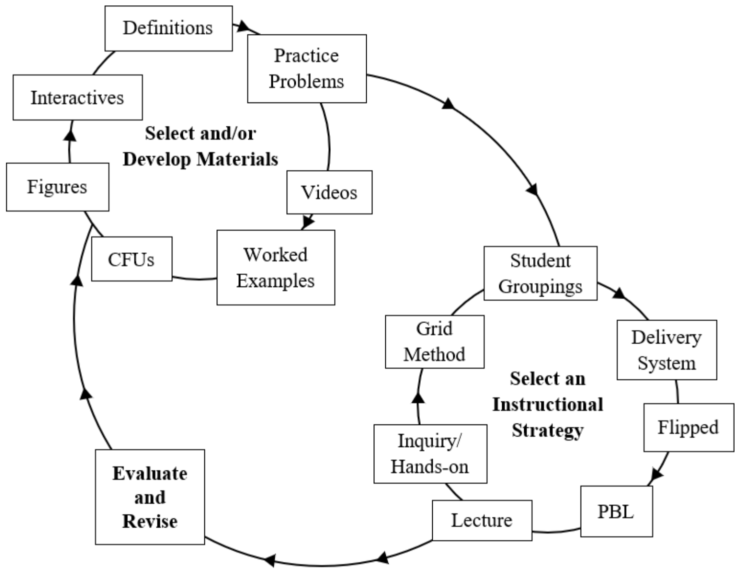
Assessment

Success or failure on the Geometry EOC exam has lasting implications for students' graduation trajectory. The Dick & Carey model suggest that instructional designers should develop assessment instruments following the development of performance objectives (Dick et al., 2015). While additional, criterion-referenced assessments are important for evaluating student progress, learners in the course will also be assessed using the Geometry EOC, which is also a norm-referenced assessment. A criterion-referenced test is designed to measure achievement “on specific goals and objectives within a given content area” (Dick et al., 2015, p. 138). A norm-referenced assessment, however, is used to compare individual’s performances to others’ performance, “describing learners’ general achievement and ability levels using terms such as *above average* (compared to their peers), *average*, or *below average* in a given subject area” (Dick et al., 2015, p. 138). Unlike the Dick & Carey model, where the instructional designer must develop an assessment instrument, for proof-writing, the summative assessment instrument has already been developed.

To ensure that all performance objectives for a unit of instruction have been identified, and the objectives are written to the appropriate level of rigor and development, it is essential that performance objectives be aligned to the Geometry EOC exam. This ensures that the materials used and skills developed are appropriate for mastery on the norm-referenced summative assessment. In order to align objectives, instructional designers must examine previously released test items as well as other Ohio Department of Education guidance to determine if performance objectives are appropriate for the given content. This is an iterative process – as the instructional designer aligns the performance objectives to the state assessment, modifications and adjustments may be necessary to improve objectives. Aligning the objectives to the appropriate level of rigor of the assessment, as well as the determining if all necessary skills have been identified, allow for the development of appropriate entry skills tests and pre-tests, which are used to assess learners’ mastery of prerequisite skills and prior knowledge, which are important in the adjustment of instruction to meet learners’ specific needs.

Figure 3

Development of Instruction Cycle in Proof-Writing Instructional Design Model



Development of Instruction

In the Proof-Writing Instructional Design Model, after the performance objectives have been aligned to the Geometry EOC, instructors now develop the instruction for the unit. Dick et al. (2015) note that there are several external factors that impact the selection and development of instructional materials and strategies, including the delivery mode, available time, and budget. Additionally, student motivation using Keller's ARCS Model should be considered for all components of instruction (Dick et al., 2015). The development of instruction (see Figure 3) is comprised of three components: select and/or develop materials; select an instructional strategy; and evaluate and revise. The three components are cyclical and can be completed in any order, with adjustments and revisions made as needed throughout the development process.

Materials

Instructional materials are "any preexisting materials being incorporated, as well as… those materials developed specifically for the objectives" (Dick et al., 2015, p. 259). In the typical design of instruction, classroom instructors select or develop materials as needed for their instructional use. This same process occurs when the instructional designer is the instructor: the development process is informal with the understanding that materials can be adapted or adjusted as needed throughout their use (Dick et al., 2015). There are a wide variety of instructional materials available for classroom use, including “student workbooks, activity guides, problem scenarios, computer simulations, case studies, resource lists, and other such materials” (Dick et al., 2015, p. 259). When selecting materials, the instructor needs to consider the learner profile, as well as how engaging materials are and how the materials work with the instructional strategy and delivery mode.

It is vital that an instructor also consider the performance objectives when selecting materials. Performance objectives direct decisions about the materials that are most effective for instruction. This is a cyclical process, however. Instructors often have multiple options from which to select or they can develop their own materials, and as the instruction is designed and the instructional strategy is determined, changes may need to be made to planned materials. Conversely, the selection of one specific material, such as a computer interactive or simulation, might dictate a specific instructional strategy, such as inquiry-based learning. Proof-writing is a developmental process; as a learner masters the postulates, theorems, and corollaries for proofs about one concept, new postulates, theorems, and corollaries are introduced with new content. In its initial iteration, proof-writing requires a certain level of interaction and collaboration between students and with the instructor for learners to receive the feedback necessary to develop the skills to successfully write a proof. However, as learners develop their proof-writing skills, the required amount of interaction with the instructor decreases as learners require less feedback and are able to solve more complex proofs.

Instructional Strategy

Dick et al. (2015) define instructional strategy as "a huge variety of teaching/learning activities such as group discussions, independent reading, case studies, lectures, computer simulations, worksheets, and cooperative group projects" (p. 173). An instructional strategy is comprised of five components: "pre-instructional activities, content presentation, learner participation, assessment, [and] follow-through activities" (Dick et al., 2015, p. 175). Pre-instructional activities are intended to motivate learners, tell the learner about the objective(s), and activate prior knowledge, while the content presentation is the new information that is introduced to a learner (Dick et al., 2015). Learner participation requires interaction by the learner with the content, either through practice with feedback or embedded practice tests before learners complete the assessment to determine what they have learned (Dick et al., 2015). The final component of an instructional strategy is the follow-through activities, where what learners remember and are able to transfer are assessed (Dick et al., 2015).

As with the selection of instructional materials, the development of the instructional strategy must be directed by the performance objectives. Dick et al. (2015) caution that "the learning components of an instructional strategy should be planned selectively rather than being provided slavishly for all learners in all instructional settings" (p. 183). Instead of incorporating every possible learning strategy for an objective, the designer must make choices about the activities they choose to include in the instruction, guided by the information collected from their earlier analyses of the content, the performance context, and the learners. In proof-writing instruction, it is important that designers incorporate learning guidance, through the use of activities such as outlining, modeling, or examples, and opportunities for feedback, which gives learners a chance to understand their progress toward the objective (Dick et al., 2015). The selection of the instructional strategy is affected by the choices a designer makes about instructional materials. Just as the use of some instructional materials dictate the instructional strategy, the converse is true: the use of some instructional strategies mandates or excludes some types of instructional materials. For example, if a designer determines that a lecture is the best instructional strategy for a specific proof-writing concept, the use of interactives may not make sense as an instructional material.

Evaluation and Revision

Evaluation and revision of materials is a key part of the instructional design process. Dick et al. (2015) argue that "the purpose of the formative evaluation is to pinpoint specific errors in the materials in order to correct them" (p. 285). For instructor-designers, this process of evaluation and revision often takes place during the design process, as indicated in the Proof-Writing Instructional Design Model, or during the execution of the lesson. While Dick et al. (2015) recommend a three-phase formative evaluation process, with clinical one-to-one evaluation, small group evaluation, and field trials, this structure is rarely possible for instructor-designers because there is neither the time nor the learners available to pre-test the instruction (Dick et al., 2015). Dick et al. (2015) suggest that for an instructor-designer, the formal formative evaluation process most closely resembles a field trial, where "the instructor should be concerned with the entry skills and prior knowledge, the posttest knowledge, and the attitudes of learners" (p. 297). Formal evaluation and revision of the instruction occur during the execution of the lesson in response to instructor-identified issues with the instructional materials or instructional strategy. When the designer is the instructor, in-the-moment adjustments are possible, with instruction easily adjusted to meet the needs of learners and the learning context.

Discussion

In most high school classrooms, the classroom instructor is also the instructional designer. While the Dick & Carey model allowed for the possibility that the instructional designer was also the instructor, certain components of the model were unrealistic for a classroom instructor. In the Dick & Carey model, four areas were potentially challenging for instructors: the evaluation process; the development of the assessment; the process of instructional material and strategy selection; and the use of formative evaluation. The Proof-Writing Instructional Design Model was developed to be used by classroom instructors in order to address these areas of challenge in the Dick & Carey model.

The Dick & Carey model assumed both that the designer did not know the target population and that the designer was unlikely to be a subject-matter expert. The Proof-Writing Instructional Design Model acknowledges that when an instructor is designing the instruction, they are the person analyzing the collected information from the needs assessment and making cognizant choices about content, learners, and the performance context based on their professional experience. As the classroom instructor who knows the target audience, a teacher is likely to have a better understanding of the learners in the classroom, as well as the entry skills that have been taught. In the Dick & Carey model, the performance objectives directed the development of assessment instrument. This, however, ignores the fact that for many subjects there are external assessments that are directing the instructional design. The Proof-Writing Instructional Design Model acknowledges that state EOC items should be the measure to which instruction and additional assessments are aligned, with instructors using previously released items and practice tests to align all aspects of instruction.

While systematic, the development of the instructional strategy and instructional materials in the Dick & Carey model required the instructional strategy to be decided before developing instructional materials, which does not necessarily reflect the reality of a classroom instructor. The new model acknowledges that the decision about strategy and materials may not always occur in the same order and may need to be revised throughout the design process as the instructor makes decisions about the instruction. An instructor may select or develop materials which align with a different instructional strategy than anticipated or a classroom instructor may be limited to specific instructional strategies by outside factors. Finally, the three-phase formative evaluation process as described by Dick & Carey is unrealistic in a classroom setting, especially due to time. Instructors often have little turn-around time between designing a lesson and administering it to students. The Proof-Writing Instructional Design Model incorporates feedback throughout the model, not as a specific, penultimate step in the design process, which allows for immediate revision throughout the process instead of after materials have been created. When possible, many instructors do modify and revise their instructional materials and strategies over the course of a day if teaching the lesson multiple times, so evaluation and revision does happen at the end of the design process, but it is often a cyclical process with changes being made as needed based on the learners' needs.

Limitations

While the Proof-Writing Instructional Design Model is intended to demonstrate the instructional design process for designing instruction of proof-writing, the content about which proofs are written covers a wide variety of geometric concepts. In order to allow for applicability to the variety of geometric concepts, the instructional design model may be overly vague. Additionally, the instructional design model is intended for instructors who are the instructional designers for their own instruction; the model is unlikely to apply if the instructional designer is not the instructor because much of the model relies on the instructor's knowledge of their students and the content itself, as well as their ability to adapt instruction as needed in-the-moment.

Conclusion

Proof-writing is a foundational component of high school Geometry instruction, incorporated in standards throughout the Geometry curriculum, beginning with parallel line proofs, continuing through triangle congruence proofs to triangle similarity proofs and quadrilateral proofs. The instructional designer for most Geometry classrooms is the classroom instructor. The Proof-Writing Instructional Design Model, which was developed from the Dick & Carey (1990) instructional design model, incorporates the unique concerns and experiences that instructors face as instructional designers, including an adapted evaluation process, alignment of the performance objectives to the assessment, and the cyclical process of the development, evaluation, and revision of instruction.

References

Bass, H. (2015) Mathematics and teaching. In P. Casazza, S. G. Krantz, R. D. Ruden (Eds.), *I, Mathematician*. (pp. 129-39). Mathematical Association of America.

Common Core State Standards Initiative. (2021). *High school: Geometry introduction*. Common Core State Standards Initiative. http://www.corestandards.org/Math/Content/HSG/introduction/.

Dick, W., Carey, L., & Carey, J. O. (2015). *The systematic design of instruction*. Pearson.

National Council of Teachers of Mathematics (NCTM). (2018). *Catalyzing change in high schools: Initiating critical conversations.* The National Council of Teachers of Mathematics, Inc.

Ohio Department of Education. (2018). *Ohio's learning standards of Mathematics: Geometry.* http://education.ohio.gov/getattachment/Topics/Learning-in-Ohio/Mathematics/Ohio-s-Learning-Standards-in-Mathematics/GEOMETRY-Standards.pdf.aspx?lang=en-US.

Ohio Department of Education. (2019a). *Math graduation curriculum requirement.* http://education.ohio.gov/getattachment/Topics/Ohios-Learning-Standards/Mathematics/Math-Graduation-Requirements/Math-Graduation-Curriculum-Requirement.pdf.aspx.

Ohio Department of Education. (2019b). *Ohio's high school graduation requirements: Classes of 2021 and 2022.* http://education.ohio.gov/getattachment/Topics/Ohio-s-Graduation-Requirements/Earning-an-Ohio-High-School-Diploma-for-the-Cl-2/GradReq2021.pdf.aspx?lang=en-US.